A Simplest Swimmer at Low Reynolds Number: Three Linked Spheres

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We propose a very simple one-dimensional swimmer consisting of three spheres that are linked by rigid rods whose lengths can change between two values. With a periodic motion in a non-reciprocal fashion, which breaks the time-reversal symmetry as well as the translational symmetry, we show that the model device can swim at low Reynolds number. This model system could be used in constructing molecular-size machines.

The usual swimming mechanism for a human being in water involves obtaining a forward momentum from the surrounding fluid due to some periodic body motion. The fact that the displacement gained in the first half period of the cyclic motion is not canceled out by that of the second half period is known to be predominantly because of the inertial effects [1]. Such a mechanism, however, does not work in the microscopic world of biological objects (such as bacteria), where the effects of inertia are not important and the viscous effects dominate. This case is characterized by very low Reynolds number—the dimensionless quantity that measures the ratio between the inertial term and the viscous term in the hydrodynamical equation of motion [2].

Most microscopic biological objects can swim very well with velocities of the order of 1 μm/s, which for such micron-sized animals swimming in water, yields Reynolds numbers of the order of 10^{-4}. In his pioneering work, Purcell showed that animals like scallop that are equipped with a single hinge cannot swim, using a simple opening and closing procedure [3, 4]. The reason is simple: since the motion is reversible, after finishing a cycle the animal will end up being where it initially was. He proposed that a non-reciprocal motion, which breaks the time-reversal symmetry, is needed to produce a net displacement. This will help the animal to propel itself, during each cycle, along some direction that is preferred by the symmetry of the system and the motion [5]. Despite the simplicity of the geometry that he suggested—three rigid rods connected with two hinges— quantitative analysis of the "Purcell swimmer" had not been performed until very recently, due to the complexity of the Stokes equation in the specific geometry [6]. We note that there are other related swimming mechanisms at the mesoscopic scale in the context of molecular machines [7].

Here we use Purcell's original idea and introduce a very simple and experimentally accessible model system that can swim using proposed periodic internal motions. The swimmer consists of three hard spheres that are linked through two arms, and has the advantage that the details of the hydrodynamic interactions, as well as the swimming velocity and direction, can be worked out with
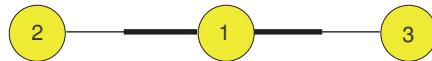
FIG. 1: Three linked spheres connected by two arms of negligible thickness.

great ease, as compared to the case of the Purcell swimmer.

The model swimmer that we are proposing is shown in Fig. 1, and consists of three spheres with radius R that are connected by rigid slender arms aligned along the x direction. The spheres are assumed to be floating in a highly viscous fluid with viscosity μ. There are two internal engines on the middle sphere (sphere number 1), which act as internal active elements responsible for making a non-reciprocal motion that is needed to propel the whole system. We consider the initial state of the system such that the spheres number 2 and 3 are in equal distance D from the middle sphere. We divide a complete cycle of the non-reciprocal motion into four parts as below (see Fig. 2):

(a) In the first step of the motion, the right arm has fixed length, and the length of the left arm is decreased with a constant relative velocity W, using one of the internal engines in the middle sphere. We denote the relative displacement of the spheres 1 and 2 in this stage by ϵ.

(b) As the second step, the left arm is fixed and the right arm decreases its length with the same constant relative velocity W as before. The relative displacement of the spheres 1 and 3 is again ϵ, like the previous stage.

(c) During this step, while the right arm is kept fixed, the left arm increase its length with the same relative velocity W to reach its original length D.

(d) Finally, in the last step the left arm is kept fixed and the right arm elongates to its original length with the same constant velocity W. The system is now in its original internal configuration.

As can be seen from Fig. 2, the above 4-stage cycle is not invariant under time reversal, and we can thus expect a net translation upon completing a full cycle. To obtain a net translational motion, the above cycle can be

FIG. 3: An auxiliary (fictitious) movement in which the right arm has a constant length δ while the left arm changes its length from D to $D - \epsilon$. During this movement the middle sphere will be displaced by an amount $\Delta_f(\delta)$.
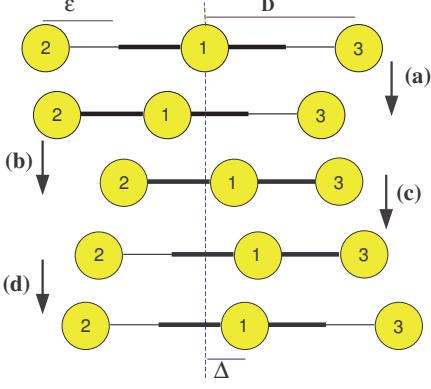
FIG. 2: Complete cycle of the proposed non-reciprocal motion of the swimmer, which is composed of 4 consecutive time-reversal breaking stages: (a) the left arm decreases its length with the constant relative velocity W, (b) the right arm decreases its length with the same velocity, (c) the left arm opens up to its original length, and finally, (d) the right arm elongates to its original size. By completing the cycle the whole system is displaced to the right side by an amount Δ.

repeated continuously.

The general equation that describe the hydrodynamics of low Reynolds number flow is the Stokes equation for the velocity field $\mathbf{u}$, subject to the incompressibility condition:

$$\mu \nabla^2 \mathbf{u} - \nabla p = 0, \tag{1}$$
$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

where p represents the pressure field in the medium.

Assuming that the spheres are moving inside the fluid with velocity vectors $\mathbf{V}_i$, with the index i denoting the labels of the spheres, the description of the system involves solving the above-mentioned equations with zero velocity boundary condition at infinity and no-slip boundary conditions on the spheres, which implies

$$\mathbf{u}|_{\mathbf{r} \text{ on the } i'\text{th sphere}} = \mathbf{V}_i. \tag{3}$$

The variables that are necessary to determine the dynamics of the spheres are their velocities $\mathbf{V}_i$ and the forces $\mathbf{F}_i$ acting on them. By solving the above governing equations, we will be able to obtain the fluid velocity in the medium, and hence, the corresponding stress tensor that will give us the required forces on the spheres. It is a simple observation that because of the linearity of the governing equations, and the linearity of the stress tensor with respect to the velocity field, one can generally expect a relation of the form:

$$\mathbf{V}_i = \sum_{j=1}^{3} \mathcal{H}_{ij} \cdot \mathbf{F}_j, \tag{4}$$

where the symmetric Oseen tensor $\mathcal{H}_{ij}$ depends on viscosity, the geometry of the bodies immersed in it (in our case the spheres), and their relative orientations. Including the condition that there are no external forces such as gravity, the system of spheres should be force-free:
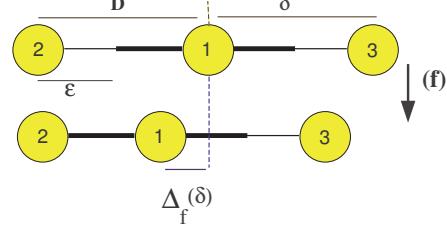
$$\sum_{j=1}^{3} \mathbf{F}_j = 0. \tag{5}$$

Since we are only interested in the dynamics of the spheres, we can equivalently solve the set of Eqs. (4) and (5), instead of Eqs. (1) and (2). To solve these equations we need to know the form of the Oseen's tensor. Let us consider a coordinate system in which the position vector of the i'th sphere is $\mathbf{x}_i$ and the separation between the i'th and the j'th spheres will be $\mathbf{x}_{ij} \equiv \mathbf{x}_i - \mathbf{x}_j$, with a unit vector $\hat{\mathbf{n}}$ in this direction. General symmetry considerations will allow us to write the hydrodynamic interaction tensor in the following form [8, 9]:

$$\mathcal{H}_{ij} = \frac{1}{6\pi\mu R} [A_{ij}(\lambda)\hat{\mathbf{n}}\hat{\mathbf{n}} + B_{ij}(\lambda)(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}})]. \tag{6}$$

where we use the dimensionless quantity $\lambda = R/x_{ij}$. Assuming that the separations between the spheres are sufficiently larger than their sizes, we can write a perturbation expansion for the symmetric coefficients A_{ij} and B_{ij} in powers of λ, which reads

$$A_{ij} = \begin{cases} 1 + O(\lambda^4), & i = j \\ \frac{3}{2}\lambda_{ij} + O(\lambda^3), & i \neq j \end{cases} \tag{7}$$

and

$$B_{ij} = \begin{cases} 1 + O(\lambda^4), & i = j \\ \frac{3}{4}\lambda_{ij} + O(\lambda^3), & i \neq j \end{cases} \tag{8}$$

to the leading order. Note that for simplicity we are considering only the translational motion for the spheres, although the effects of rotational motion can be taken into account in a similar way. Moreover, extra simplification comes from the one-dimensional nature of the motion, such that the tensorial structure of the hydrodynamic interactions plays no important role in the dynamics.

To analyze the motion of the system during one complete period of the non-reciprocal cycle, we introduce an

auxiliary move as shown in Fig. 3. During this motion, the right arm has a constant length δ while the left arm changes its length from D to $D - \epsilon$ with the constant velocity W. Using symmetry arguments, we can relate all the four steps in the non-reciprocal cycle to the above move, as follows: The step (a) corresponds to the auxiliary move by setting $\delta = D$. By applying a reflection transformation with respect to any point on the x axis, the step (b) can be mapped onto the auxiliary move with $\delta = D - \epsilon$. The step (c) is obtained by applying a time-reversal transformation on the auxiliary move, with $\delta = D - \epsilon$. Finally, the step (d) is obtained by applying a reflection transformation [as in step (b)] followed by a time-reversal transformation, with $\delta = D$.

To obtain the net displacement of the middle sphere in the real problem, it is thus enough to solve the dynamical equation for a single auxiliary movement. If we define the net displacement of the middle sphere during the auxiliary step by $\Delta_{\mathrm{f}}(\delta)$, then by considering the above arguments we can calculate the total displacement Δ of the real system through a complete cycle as

$$\Delta = 2[\Delta_{\mathrm{f}}(D) - \Delta_{\mathrm{f}}(D - \epsilon)]. \tag{9}$$

Noting that the above displacement takes place during the time of a complete cycle that is $4\epsilon/W$, then we can calculate the average swimming velocity for the swimmer as

$$V_s = W \frac{\Delta}{4\epsilon}. \tag{10}$$

As an example, we have numerically calculated the displacements for the case of $D = 10\,R$ and $\epsilon = 4\,R$. During the first stroke the middle sphere swims in the $-x$ direction by an amount $1.35\,R$, while in the second stroke it swims a distance of $1.44\,R$ in the positive direction. The third stroke then causes a continuation of the motion in the positive direction for a distance of $1.44\,R$, and finally, the fourth stroke takes it back by a distance of $1.35\,R$. At the end of this cycle the sphere is displaced by a net amount of $0.16\,R$ in the positive direction, as shown in Fig. 2. Figure 4 shows the total displacement of the middle sphere during a complete non-reciprocal cycle as a function of the internal relative displacement ϵ for $D = 10\,R$.

In the limit of small internal deformation of the swimmer, we can expand all the quantities in terms of ϵ/D, and calculate the swimming velocity in a perturbative series. To the leading order, we find

$$V_s = 0.7\,W \left(\frac{R}{D}\right) \left(\frac{\epsilon}{D}\right)^2. \tag{11}$$

The above result shows that the scale of the swimming velocity is set by the typical velocity of the internal motion. Moreover, the swimming appears to be a quadratic effect with respect to the small deformations in the system. These two characteristics are general, as can be seen in other swimmers at low Reynolds number.
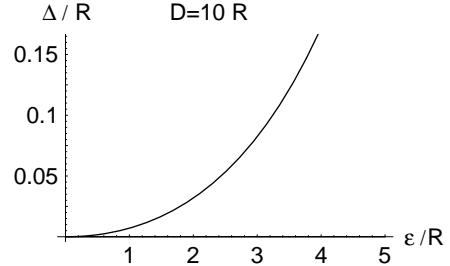

FIG. 4: Dimensionless displacement of the swimmer in a complete cycle as a function of the dimensionless relative displacement between neighboring spheres.

In our simple prescription for the non-reciprocal motion we have assumed that in each step one sphere moves with respect to the middle one and the other one is kept in relative constant distance. This assumption has been made for simplicity, and we can imagine a more general continuous motion of the spheres with respect to the middle one. The only necessary condition is to break the time-reversal symmetry and obtain a non-reciprocal motion. The general requirement in the continuous motion will be the existence of a nonzero phase difference between the continuous periodic motions of the left and right spheres with respect to the middle one. The swimming velocity in this case will also have the two general characteristics mentioned above.

The generalized case of many spheres that are coupled to each others on a regular one- or two-dimensional lattice is a suitable microscopic model for an extensible flat filament or membrane. In these cases, the relative "in-plane" motion of the neighboring spheres with respect to each other will cause the whole system to swim. The direction of motion and the swimming velocity can be predicted using the simple proposed description. The internal relative motion of the system that can cause swimming is of the general form of a traveling wave on the position of the spheres in some direction. Traveling wave is the simplest non-reciprocal motion which breaks the time-reversal and translational symmetries. The swimming velocity will be opposite to the direction of the wave-vector, and proportional to the phase velocity of the traveling wave. Such kind of motion has been considered for a spherical membrane, and shown to have the same characteristics as suggested above [10].

In conclusion we have introduced a very simple swimmer and calculated its swimming velocity. The swimmer uses some periodic internal motion to propel itself under low Reynolds number conditions. The advantage of this model, as compared to previously known model swimmers, is that the analysis of the hydrodynamics problem can be performed considerably more easily. The model swimmer could be used in making molecular-size machines with controllable motion.

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